MATH 31B, LECTURE 1
FINAL EXAMINATION
JUNE 15, 2012

Name:____

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Signature: ____

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Discussion meets: (circle one) Tuesday Thursday

Instructions: The exam is closed-book, closed-notes. Calculators are not permitted. Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 10 points.

Problem	Score
1	
2	
3	
4	
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6	
7	
8	
9	
10	
Total	

Problem 1: (Multiple Choice). Indicate your answer in the provided box.

(i) Evaluate the infinite series: $\sum_{n=0}^{\infty} \frac{1+2^n}{4^{n+1}}$. $\boxed{\text{c}}$

(a) 1 (b) 16/3 (c) 5/6 (d) 10/3 (e) 4/3

(ii) Evaluate the infinite series: $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$. $\boxed{\text{a}}$

(a) 3/2 (b) 2 (c) 1/2 (d) 3/4 (e) 1

(iii) Evaluate the infinite series: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. $\boxed{\text{d}}$

(a) $\pi/2$ (b) e^{-1} (c) 1/2 (d) $\pi/4$ (e) 3/4

(iv) Evaluate the limit: $\lim_{n\to\infty} \sqrt[n]{n^2+1}$. $\boxed{\text{d}}$

(a) 1/2 (b) $+\infty$ (c) 0 (d) 1 (e) 2

(v) Which diff. equation does $y = 1 + \frac{1 \cdot x^3}{3!} + \frac{4 \cdot 1 \cdot x^6}{6!} + \frac{7 \cdot 4 \cdot 1}{9!}x^9 + \frac{10 \cdot 7 \cdot 4 \cdot 1}{12!}x^{12} + \cdots$ satisfy? $\boxed{\text{b}}$

(a) $y' = x^2 y$ (b) $y'' = xy$ (c) $y'' + xy' - x^2 y = 0$ (d) $y'' = y$ (e) None of these

Problem 2: (Multiple choice). Indicate the Maclaurin series (a-h) corresponding to $f(x)$.

(i) $f(x) = \frac{2x}{2+x^2}$. $\boxed{\text{h}}$

(ii) $f(x) = x\ln(1+x)$. $\boxed{\text{d}}$

(iii) $f(x) = \cos x^2$. $\boxed{\text{a}}$

(iv) $f(x) = \tan^{-1} x$. $\boxed{\text{g}}$

(v) $f(x) = \sin x \cos x$. $\boxed{\text{e}}$

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!}$ (b) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n}$ (c) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$ (d) $\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{n-1}$

(e) $\sum_{n=0}^{\infty}(-1)^n \frac{4^n \cdot x^{2n+1}}{(2n+1)!}$ (f) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}$ (g) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ (h) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2^n}$

Problem 3: Evaluate the limits.

(a) $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}+\dfrac{1}{1-x}\right)$

(b) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+2}\right)^x$

Solution:

(a) We have

$$\lim_{x\to 1}\left(\frac{1}{\ln x}+\frac{1}{1-x}\right)=\lim_{x\to 1}\frac{1-x+\ln x}{(1-x)\ln x}\overset{L.H.}{=}\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\ln x+\frac{1-x}{x}}=\lim_{x\to 1}\frac{-x+1}{-x\ln x+1-x}$$
$$\overset{L.H.}{=}\lim_{x\to 1}\frac{-1}{-\ln x-1-1}=\frac{1}{2}.$$

(b) We have

$$\lim_{x\to\infty}\left(\frac{x}{x+2}\right)^x=\lim_{x\to\infty}\left(1-\frac{2}{x+2}\right)^x=\lim_{x\to\infty}e^{x\ln\left(1-\frac{2}{x+2}\right)}.$$

Now compute

$$\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)}\overset{L.H.}{=}\lim_{x\to\infty}\frac{\frac{1}{1-\frac{2}{x+2}}\cdot\frac{2}{(x+2)^2}}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}-\frac{1}{1-\frac{2}{x+2}}\cdot\frac{2x^2}{(x+2)^2}=-2.$$

Plugging in above we find $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+2}\right)^x=e^{-2}$.

Problem 4: Let $f(x)=(\cos x)^{\tan x}$, find $f'(x)$.

Solution: We have $f(x)=(\cos x)^{\tan x}=e^{\tan x\ln(\cos x)}$, so

$$f'(x)=\left[\sec^2 x\ln(\cos x)-\tan x\cdot\tan x\right]e^{\tan x\ln(\cos x)}=\left[\sec^2 x\ln(\cos x)-\tan^2 x\right](\cos x)^{\tan x}.$$

Problem 5:

(a) Evaluate the indefinite integral $\displaystyle\int\sin^3(x)\,dx$.

(b) Evaluate $\displaystyle\lim_{R\to\infty}\int_{-R}^{R}\sin^3(x)\,dx$.

(c) Evaluate the improper integral $\displaystyle\int_{-\infty}^{\infty}\sin^3(x)\,dx$, or show that it diverges.

Solution:

(a) We have

$$\int\sin^3 x\,dx=\int(1-\cos^2 x)\sin x\,dx=-\int(1-u^2)\,du=\frac{u^3}{3}-u+C=\frac{\cos^3 x}{3}-\cos x+C,$$

where we have made the substitution $u=\cos x$.

(b) We have

$$\lim_{R\to\infty}\int_{-R}^{R}\sin^3 x\,dx=\lim_{R\to\infty}\left(\frac{\cos^3 x}{3}-\cos x\right)\Bigg|_{x=-R}^{R}=\lim_{R\to\infty}\frac{\cos^3 R}{3}-\cos R-\frac{\cos^3(-R)}{3}+\cos(-R)=0,$$

since we have $\cos(-R)=\cos R$ for any R.

(c) We have

$$\int_{-\infty}^{\infty} \sin^3 x\, dx = \int_{-\infty}^{0} \sin^3 x\, dx + \int_{0}^{\infty} \sin^3 x\, dx.$$

The second term is

$$\lim_{R\to\infty} \int_0^R \sin^3 x\, dx = \lim_{R\to\infty} \left(\frac{\cos^3 x}{3} - \cos x\right)\Big|_{x=0}^{R} = \lim_{R\to\infty} \frac{\cos^3 R}{3} - \cos R + \frac{1}{3} - 1.$$

$f(R) = \frac{\cos^3 R}{3} - \cos R$ does not have a limit as $R \to \infty$ (for example since $f(2n\pi) = -2/3$ while $f((2n+1)\pi) = 2/3$ for all $n \geq 0$), so this integral diverges and therefore $\int_{-\infty}^{\infty} \sin^3(x)\, dx$ diverges as well.

Problem 6: Evaluate the integral: $\int_{-\infty}^{0} e^x \sqrt{1-e^{2x}}\, dx.$

Solution: First make the substitution $u = e^x$, $du = e^x\, dx$:

$$\int_{-\infty}^{0} e^x \sqrt{1-e^{2x}} = \int_0^1 \sqrt{1-u^2}\, du.$$

Now substitue $u = \sin\theta$, $du = \cos\theta\, d\theta$:

$$\int_0^1 \sqrt{1-u^2}\, du = \int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \tfrac{1}{2}(1+\cos 2\theta)\, d\theta = \tfrac{1}{2}\big(x + \tfrac{\sin 2\theta}{2}\big)\big|_0^{\pi/2} = \tfrac{1}{2}\big(\pi/2 + 0\big) - 0 = \frac{\pi}{4}.$$

Problem 7:

(a) Find the partial fractions decomposition of $\dfrac{2}{x^3 - 2x^2 + 2x}$.

(b) Evaluate the indefinite integral: $\displaystyle\int \frac{2}{x^3-2x^2+2x}\, dx.$

Solution:

(a) The partial fractions decomposition is

$$\frac{2}{x^3-2x^2+2x} = \frac{2}{x(x^2-2x+2)} = \frac{A}{x} + \frac{Bx+C}{x^2-2x+2}.$$

Note $x^2 - 2x + 2$ is irreducible since $(-2)^2 - 4(1)(2) = -4 < 0$. So we need

$$A(x^2-2x+2) + Bx^2 + Cx = 2 \Rightarrow A = 1, B = -1, C = 2$$

So we find

$$\frac{2}{x^3-2x^2+2x} = \frac{2}{x(x^2-2x+2)} = \frac{1}{x} + \frac{2-x}{x^2-2x+2}$$

(b)By part (a) we have

$$\int \frac{2}{x^3-2x^2+2x}\, dx = \int \frac{dx}{x} + \int \frac{2-x}{x^2-2x+2}\, dx$$

For the second integral complete the square $x^2 - 2x + 2 = (x-1)^2 + 1$ and substitute $u = x - 1$:

$$\int \frac{2-x}{x^2-2x+2}\, dx = \int \frac{1-u}{u^2+1}\, du = \int \frac{1}{u^2+1}\, du - \int \frac{u}{u^2+1}$$
$$= \tan^{-1}(u) - \tfrac{1}{2}\ln(u^2+1) + C = \tan^{-1}(x-1) - \tfrac{1}{2}\ln\big|(x-1)^2+1\big| + C$$

Combining with the formula above we find:

$$\int \frac{2}{x^3 - 2x^2 + 2x}\, dx = \ln|x| + \tan^{-1}(x-1) - \tfrac{1}{2}\ln\left|x^2 - 2x + 2\right| + C.$$

Problem 8: Determine whether the given series converges absolutely, converges conditionally or diverges. You must carefully justify your answer.

(a) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right)$.

(b) $\displaystyle\sum_{n=2}^{\infty}(-1)^n \frac{n^2}{4^n - 2^n - n + 1}$.

Solution:

(a) First we check absolute convergence by using the limit comparison test with $a_n = \sin\left(\frac{\pi}{n}\right)$, $b_n = \frac{1}{n}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{x\to 0}\frac{\sin \pi x}{x} \overset{L.H.}{=} \lim_{x\to 0}\frac{\pi\cos\pi x}{1} = \pi.$$

By the limit comparison test, it follows that $\sum a_n$ converges if and only if $\sum b_n$ converges, but $\sum b_n$ is the harmonic series which we know to be divergent. So the series does not converge absolutely. Now let $f(x) = \sin\left(\frac{\pi}{x}\right)$. We have $f'(x) = \frac{-\pi}{x^2}\cos\left(\frac{\pi}{x}\right) < 0$ for $x > 2$. So $f(x)$ is decreasing and $\lim\limits_{x\to\infty} f(x) = \sin 0 = 0$. By the Leibniz test, the alternating series

$$\sum_{n=2}^{\infty}(-1)^n \sin\left(\frac{\pi}{n}\right) = \sum_{n=2}^{\infty}(-1)^n f(n)$$

converges.

(b) Check absolute convergence by comparing $a_n = \frac{n^2}{4^n - 2^n - n + 1}$ with $b_n = \frac{n^2}{4^n}$. We have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2}{4^n - 2^n - n + 1}\frac{4^n}{n^2} = \lim_{n\to\infty}\frac{4^n}{4^n - 2^n - n + 1} = \lim_{n\to\infty}\frac{1}{1 - \frac{1}{2^n} - \frac{n}{4^n} + \frac{1}{4^n}} = 1.$$

It follows that $\sum a_n$ converges if and only if $\sum b_n$ converges. To determine if $\sum b_n$ converges, apply the ratio test:

$$\lim_{n\to\infty}\frac{b_{n+1}}{b_n} = \lim_{n\to\infty}\frac{(n+1)^2}{4^{n+1}}\frac{4^n}{n^2} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{4} = \frac{1}{4},$$

so $\sum b_n$ converges by the ratio test and hence $\sum a_n$ converges. So the series is absolutely convergent.

Problem 9: Determine the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{(x-2)^n}{2^n + n^2}$.

Solution: Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{2^{n+1} + (n+1)^2}\cdot\frac{2^n + n^2}{(x-2)^n}\right| = |x-2|\cdot\lim_{n\to\infty}\frac{2^n + n^2}{2^{n+1} + (n+1)^2} = |x-2|\cdot\lim_{n\to\infty}\frac{1 + \frac{n^2}{2^n}}{2 + \frac{(n+1)^2}{2^n}} = |x-2|\cdot\tfrac{1}{2}$$

By the ratio test this converges if $|x-2|\cdot\frac{1}{2} < 1$ or $|x-2| < 2$, and diverges if $|x-2| > 2$. So the radius of convergence is 2.

Now check the endpoints: $2 - 2 = 0$ and $2 + 2 = 4$. At $x = 2$ we have

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n + n^2}$$

which diverges since $\lim_{n\to\infty} \dfrac{2^n}{2^n + n^2} = 1 \neq 0$. Likewise at $x = 0$ we have

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^n}{2^n + n^2}$$

which diverges since the terms do not tend to 0. So the interval of convergence is $(0, 4)$.

Problem 10: Let $f(x) = (1 + x^2)\tan^{-1}(x)$.
(a) Find the Maclaurin series for $f(x)$.
(b) Find $f^{(9)}(0)$.

Solution:
(a) We have

$$\begin{aligned}(1 + x^2)\tan^{-1}(x) &= (1 + x^2)\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+3}}{2n+1} \\ &= x + \sum_{n=0}^{\infty}(-1)^{(n+1)} \frac{x^{(2n+3)}}{2n+3} + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+3}}{2n+1} = x + \sum_{n=0}^{\infty}(-1)^n \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)x^{2n+3} \\ &= x + \sum_{n=0}^{\infty}(-1)^n \frac{2}{(2n+1)(2n+3)}x^{2n+3}\end{aligned}$$

(b) The coefficient on x^9 in the series from (a) is $(-1)^3 \frac{2}{(2\cdot 3+1)(2\cdot 3+3)} = \frac{-2}{63}$. But this coefficient must be equal to $\frac{f^{(9)}(0)}{9!}$, so it follows that $f^{(9)}(0) = \frac{-2\cdot 9!}{63}$.